Universal and nonuniversal properties of wave-chaotic scattering systems

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Prediction of the statistics of scattering in typical wave-chaotic systems requires combining system-specific information with universal aspects of chaotic scattering as described by random matrix theory. This Rapid Communication shows that the average impedance matrix, which characterizes such system-specific properties, can be semiclassically calculated in terms of ray trajectories between ports. Theoretical predictions are compared with experimental results for a microwave billiard, demonstrating that the theory successfully uncovered universal statistics of wave-chaotic scattering systems.

Wave systems appear in diverse branches of physics, such as quantum mechanics, electromagnetics, and acoustics. However, solving the wave equations can be difficult, particularly in the short-wavelength limit [1]. Furthermore, even if exact solutions were feasible, there may be uncertainties in the locations of boundaries or in parameters specifying the system, and the desired wave quantities can be extremely sensitive to such uncertainties when the wavelength is short. Thus, rather than seeking solutions for specific systems, it is often convenient to create statistical models which reproduce generic properties of the system [2]. This is the motivation for the application of random matrix theory to wave-chaotic systems, in which it is conjectured that useful statistics can be obtained by replacing the exact Hamiltonian or scattering matrices by random matrices drawn from an appropriate ensemble. Here, by wave chaotic we mean that, in the small wavelength limit, the behavior of the wave system is described by ray orbit trajectories that are chaotic [3]. Although such formulations cannot predict details of any particular wave system, they do predict the distribution of properties in an ensemble of related wave-chaotic systems. Random matrix theory is also hypothesized to predict the statistical properties of a *single* wave-chaotic system evaluated at different frequencies (in, e.g., the cases of acoustics or electromagnetics) or energies (in the case of quantum mechanics). See Refs. [4–6] for reviews of the theory, history, and the wide range of applications of random matrix theory.

In this Rapid Communication, random matrix theory is applied to model the scattering behavior of an ensemble of wave-chaotic systems coupled to the outside world through a single scattering channel (the generalization to larger numbers of scattering channels is straightforward [7]). Such scattering systems have been studied extensively [8–13], with most work focusing on the scattering parameter S, which is the ratio between the reflected waves and the incident waves in the channel. Here we consider ensembles of systems whose distribution of scattering parameters are well described by the so-called Poisson kernel [8–10,14]. The Poisson kernel characterizes the probability density for observing a particular scattering parameter S in terms of the average scattering parameter $\bar{S}$. It represents contributions to the scattering behavior from elements of the system which are not random, such as the prompt reflection from the interface be-tween the scattering channel and the chaotic system. In addition, rays within the scattering region which return to the scattering channel without ergodically exploring the chaotic dynamics also affect $\bar{S}$ [11,15,16]. The ability to determine $\bar{S}$ from first principles, thus incorporating all nonuniversal effects, would dramatically improve our understanding and ability to uncover universal fluctuations in a host of wave phenomena. Because $\bar{S}$ is the only parameter in the Poisson kernel, methods for finding it are of interest. Even though $\bar{S}$ can be estimated from experimental ensemble data, *predicting* it from first principles has so far not been addressed in general (although it has been done for the specific case of quantum graphs [17]). In this Rapid Communication we show how to semiclassically obtain $\bar{S}$, and we experimentally verify the accuracy and utility of our technique.

A quantity equivalent to the scattering parameter S is the impedance, $Z = Z_0(1+S)/(1-S)$, where Z_0 is the characteristic impedance of the scattering channel. Because nonuniversal contributions manifest themselves in Z as simple additive corrections, we use Z in much of our discussion [13,18–20]. We note that impedance is a meaningful concept for all scattering wave systems. In linear electromagnetics, it is defined via Ohm's law as $\hat{V} = Z\hat{I}$, where $\hat{V}$ represents the phasor voltage difference across the attached transmission line (the system's port) and $\hat{I}$ denotes the phasor current flowing through the transmission line. In acoustics, the impedance is the ratio of the sound pressure to the fluid velocity. A quantum-mechanical quantity corresponding to impedance is the reaction matrix [5,13,18]. In what follows, our discussion will use language appropriate to scattering from a microwave cavity excited by a small antenna fed by a transmission line (the setting for our experiments).

With the transformation to impedance, if S is distributed according to the Poisson kernel, we find that the impedance can be represented as [7]

$$Z = iX_{avg} + R_{avg}i\xi, \qquad (1)$$

where in the lossless case R_{avg} and X_{avg} are the real and imaginary parts of the impedance Z_{avg} based on the average scattering parameter, where $Z_{avg} \equiv Z_0(1+\bar{S})/(1-\bar{S})$ and $i\xi$ (which we call the normalized impedance) is a Lorentzian

distributed random variable with median 0 and width 1. With uniform loss (e.g., due to an imaginary part of a homogeneous dielectric constant in a microwave cavity), R_{avg} and X_{avg} are the analytic continuations of the real and imaginary parts of Z_{avg} as $k \rightarrow k + ik/(2Q)$, where $Q \gg 1$ is the quality factor of the closed system and k denotes the wave number of a plane wave. The normalized impedance $i\xi$ of the lossy system has a universal distribution which is dependent only on the ratio $k/(2Q\Delta k)$, where Δk is the mean spacing between modes [18,20].

We find that Z_{avg} can be evaluated directly in the semiclassical limit [7] as

$$Z_{avg} = Z_R + R_R \sum_{b(l)} \left[P_{b(l)} \sqrt{D_{b(l)}} e^{ik(l+L_{port})-i\pi/4} \right], \quad (2)$$

where the radiation impedance $Z_R = R_R + iX_R$ represents the impedance the system would have if all the energy which coupled into the system was absorbed before coupling back out [19,20]. The radiation resistance R_R and the radiation reactance X_R are the real and imaginary parts of the radiation impedance Z_R. In the summation, $b(l)$ is an index over all classical trajectories which leave the port and return to the port location with total path length l, k is the wave number, and L_{port} is observed to be a port-dependent constant length. Each term of the sum is weighted by the survival probability of the trajectory in the ensemble ($P_{b(l)}$, to be discussed further) and the geometrical factor of the trajectory ($D_{b(l)}$) [7]. This geometrical factor is a function of the length of each segment of the trajectory, the angle of incidence of each bounce, and the radius of curvature of each wall encountered in that trajectory; it has been assumed that the port radiates isotropically from a location far from the two-dimensional cavity boundaries. These parameters can all be determined from the geometry of the scatterer and location of the port [7].

The purpose of this Rapid Communication is to test the accuracy and usefulness of Eq. (2). In practice, we take account of a finite number of ray trajectories according to their length l. Therefore, in Eq. (2) we replace the summation by $\sum_{b(l),L}$ which signifies that the sum is now over all trajectories $b(l)$ with lengths up to L, $l \leq L$.

In order to verify the theory of Hart *et al.* [7], experimental tests are carried out on a quasi-two-dimensional microwave cavity with a single port [3,21] (see Fig. 1, inset), where the length of wall A is 21.6 cm, the length of wall B is 43.2 cm, the distance of the port to the nearest wall (wall D) is 7.5 cm, and the height of the cavity is 0.8 cm. For the frequency range explored (6–18 GHz), higher-order vertical modes are beyond cutoff so that the waves are quasi-two-dimensional. Furthermore, the corresponding wavelengths (1.7–5.0 cm) can be regarded as in the semiclassical regime, and the cavity shape yields chaotic ray trajectories. We excite the cavity by means of a single coaxial probe whose exposed inner conductor extends from the top plate and almost makes electrical contact with the bottom plate of the cavity [3].

The radiation impedance Z_R in Eq. (2) is measured by covering the four side walls of the cavity with microwave absorbers. Normalizing the measured impedance with the radiation impedance has been used to remove the nonuniversal
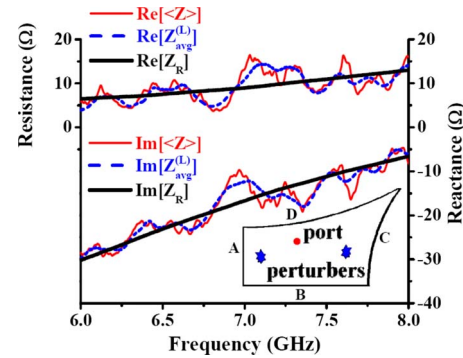

FIG. 1. (Color online) Plot of the impedance from the average of 100 cavity realizations $\langle Z \rangle$ versus frequency from 6 to 8 GHz with perturbers inside the cavity. Shown are the real (three upper curves) and the imaginary parts (three lower curves) of the impedance for the theory ($Z_{avg}^{(L)}$ with $L = 200$ cm, blue dashed) and the experiment ($\langle Z \rangle$, red solid), as well as the measured radiation impedance of the port (Z_R, black thick). Inset: The wave-chaotic two-dimensional cavity with perturbers and a single port.

properties due to the coupling of the port and the cavity [3,21,22]. Here we further consider the nonuniversal properties due to short ray trajectories by adding the summation term in Eq. (2).

To verify that Eq. (2) describes nonuniversal characteristics of wave-chaotic systems, we first proceed to determine universal statistics by applying the ensemble average. Two irregular-shaped pieces of metal (with the maximum diameters 7.9 and 9.5 cm) are added as perturbers in the wave-chaotic system that is shown in the inset of Fig. 1, where the circular dot shows the port and the two starlike objects represent the perturbers. The locations of the two perturbers inside the cavity are systematically changed to produce a set of 100 realizations for the ensemble [21]. Typically, the shifts of resonances between two realizations are about one mean level spacing. The scattering parameter S is measured from 6 to 18 GHz, covering roughly 1070 modes of the cavity. After the ensemble average, longer ray trajectories have higher probability of being blocked by the two perturbers, and therefore, the main nonuniversal contributions are due to short ray trajectories. We compare the measured ensemble averaged impedance $\langle Z \rangle$ and the theoretical impedance $Z_{avg}^{(L)}$ that is calculated from Eq. (2) with the sum up to the maximum trajectory length $L = 200$ cm, corresponding to a total of 584 trajectories. We multiply each term in the sum [Eq. (2)] by a weight ($P_{b(l)}$) equal to the fraction of perturbation configurations that are not intercepted by the trajectory corresponding to that term. The result is shown in Fig. 1 where the three upper curves are the real part of the impedance (resistance) and the three lower curves are the imaginary part (reactance). The experiment curves (red solid) follow the trend of the radiation impedance curves (black thick), and the theory curves (blue dashed) reproduce most of the fluctuations in the experiment curves with only a modest number of trajectories. The good agreement between the measured data and the theoretical prediction verifies that the new theory [Eq. (2)] predicts the nonuniversal features embodied in the ensemble averaged impedance. We believe that the
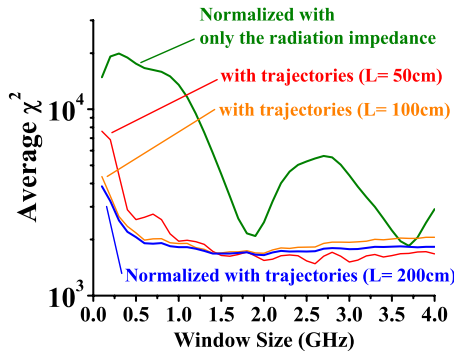
FIG. 2. (Color online) The average χ^2 of distributions of the phase of the scattering parameter φ_s on a semilogarithmic scale, where the scattering parameters are calculated from impedance normalized with only the radiation impedance (green) and with ray trajectories according to the maximum trajectory length from L $=50$ cm (red) up to $L=200$ cm (blue) versus frequency window sizes from 0.1 to 4.0 GHz.

differences between the measured data and the theory are due to the finite number of realizations in the ensemble and diffraction effects that are not taken into account in the theory.

We further test our theory by consideration of the statistics of the scattering parameter for an ensemble of perturbation configurations and an ensemble of frequencies. Random matrix theory predicts that, after all nonuniversal effects have been removed, the resulting scattering parameter, which we denote as $s=(i\xi-1)/(i\xi+1)=|s|e^{i\varphi_s}$, should be distributed uniformly in phase in $0\leq\varphi_s\leq2\pi$, and this result is independent of loss, frequency, and mean level spacing [9,20]. The previous work of Refs. [3,21,22] removed the nonuniversal properties by performing normalization with only the radiation impedance as $i\xi=(Z-iX_R)/R_R$. Here we add ray trajectory corrections based on the maximum trajectory length L,

$$i\xi^{(L)} = (Z - iX_{avg}^{(L)})/R_{avg}^{(L)}, \qquad (3)$$

and use the χ^2 test to evaluate how uniform the resulting phase distributions are. $R_{avg}^{(L)}$ and $X_{avg}^{(L)}$ are the analytic continuations of the real and imaginary parts of $Z_{avg}^{(L)}$ as $k\to k$ $+ik/(2Q)$ in the experimental case with loss. Experimental distributions of the phase φ_s of s are calculated from 100 realizations and different frequency windows. χ^2 $=\frac{1}{\langle n_i\rangle}\Sigma_{i=1}^N(n_i-\langle n_i\rangle)^2$ measures the deviation between the experimental distributions of φ_s and a perfectly uniform distribution, where n_i is the number of elements in the ith bin in the histogram (with ten bins, $N=10$) of the probability of the phase of the scattering parameter $P(\varphi_s)$ and $\langle n_i\rangle$ is the average of n_i over i. A smaller χ^2 value means the experimental data are closer to the theoretical prediction.

Figure 2 shows the averaged χ^2 evaluated over the spectral range from 6 to 18 GHz. The results indicate that the distributions of the measured data are systematically more uniform as more ray trajectories are taken into account in the impedance normalization [Eq. (3)]. The improvement is dramatic after including just a few short ray trajectories (L $=50$ cm, seven trajectories) and saturates beyond L
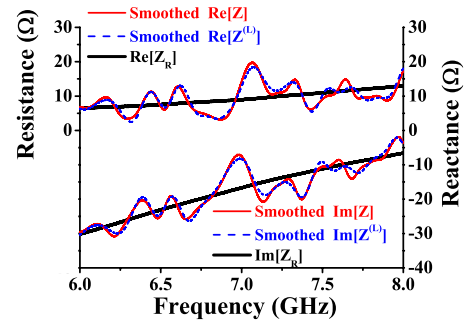

FIG. 3. (Color online) Plot of the smoothed impedance versus frequency from 6 to 8 GHz. Shown are the real (three upper curves) and the imaginary part (three lower curves) of the smoothed impedance for the theory ($Z^{(L)}$ with $L=200$ cm, blue dashed) and the experiment (red solid), as well as the measured radiation impedance of the port (Z_R, black thick).

$=100$ cm (36 trajectories). The periodic wiggles represent the effects of the strongest remaining trajectory not taken into account in the theory. Thus, we see that nonuniversal effects of ray trajectories in the ensemble of wave-chaotic systems can be efficiently removed by considering a few short ray trajectories or by increasing the window size for the frequency ensemble.

In addition to experiments with ensemble averaging over perturber positions, we now examine the theory in the stringent case of just a single realization without scatterers, and we use only a frequency ensemble. We consider frequency smoothed experimental data and compare it with the smoothed theoretical prediction. Figure 3 shows that the smoothed measured impedance Z (red solid) agrees with the smoothed theoretical impedance $Z^{(L)}$ (blue dashed). Notice that $Z_{avg}^{(L)}\to Z^{(L)}$ because there is only a single realization now. The smoothing function is a Gaussian with standard deviation $\sigma=240$ MHz, which inserts an effective low-pass Gaussian filter on the trajectory length, thus limiting the terms in Eq. (2) to those with a path length $l\lesssim c/\sigma$ $=125$ cm. The results in Fig. 3 shows that the theory correctly captures systematic contributions from short trajectories.

In conclusion, the nonuniversal effects of coupling and short ray trajectories on wave scattering in chaotic systems are predicted by a recently developed theory [7] and verified experimentally. This is accomplished through statistical tests of the scattering parameter and comparisons of impedance in an ensemble of perturbed systems, as well as a single-realization wave-chaotic system. In particular, nonuniversal effects have been better represented and removed from measured data to reveal underlying universal fluctuations in the scattering parameter. These results should be useful in many fields, such as nuclear scattering, atomic physics, quantum transport in condensed matter systems, electromagnetics, acoustics, geophysics, etc.

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